

# Representations of Integers as Sums of Squares

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

If  $s$  is a positive integer, then let  $r(s; n)$  denote the number of representations of a non-negative integer  $n$  as a sum of  $s$  integer squares. If  $\Theta(z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$  ( $q := e^{2\pi iz}$  throughout), then

$$\sum_{n=0}^{\infty} (-1)^n r(s; n) q^n = \Theta(z)^s. \quad (1.1)$$

For small  $s$ , there are well-known formulas such as Jacobi's four-squares theorem:

$$\sum_{n=0}^{\infty} (-1)^n r(4; n) q^n = 1 + 8 \sum_{n=1}^{\infty} \sum_{\substack{d|n, \\ 4 \nmid d}} (-1)^n d q^n.$$

The general problem of determining exact formulas for  $r(s; n)$  is classical in number theory. One may consult the popular book by Grosswald [1] for a thorough account (as of the early 1980s) of the subject complete with references. The series  $\Theta(z)^s$  is a modular form, and so there are abstract formulas for  $r(s; n)$  as the Fourier coefficients of modular forms. Specifically, it is well known that  $\Theta(z)^s = E_s^*(z) + c_s(z)$ , where  $E_s^*(z)$  is an Eisenstein series with explicit coefficients and  $c_s(z)$  is a cusp form. Using this fact, one may deduce asymptotic information for  $r(s; n)$ . Rankin proved [5] that  $c_s(z)$  is non-trivial for every  $s > 8$ . Therefore, the problem of computing non-trivial formulas for  $r(s; n)$  remains since the coefficients of cusp forms, although small, rarely have simple descriptions.

In a startling turnabout, Milne [3] announced formulas for  $r(4s^2; n)$  and  $r(4s^2 + 4s; n)$  for every  $s$ . His formulas were obtained by combining a variety of methods and observations from the theory of elliptic functions, continued fractions, Lie algebras, Schur functions, and hypergeometric functions. The proofs of his formulas appear in [4].

Also in [4], he proves (via similar methods) conjectures of Kac and Wakimoto on the number of representations of positive integers as sums of triangular numbers. These conjectures were born out of observations arising in the theory of Lie algebras. In a recent paper [6], Zagier also proves these conjectures. His method involves an elegant and surprisingly simple argument. Zagier notices that the generating functions in the Kac and Wakimoto Conjectures are modular forms on  $\Gamma_0(2)$  whose zeros are supported on the cusp at infinity. Two forms sharing this property with the same weight must be multiples of each other. Zagier then observes that the ‘specializations’ of suitable polynomials with certain Eisenstein series yield such forms. Therefore, these specializations equal the relevant generating functions up to easily computable constants.

For  $r(4s^2; n)$  and  $r(4s^2 + 4s; n)$ , it turns out that a similar analysis applies. The powers of  $\Theta(z)$  are modular forms on  $\Gamma_0(2)$  whose zeros are supported at the cusp inequivalent to infinity. Arguing as above with  $E^\pm(k; z)$  (see (1.5) and (1.6)) and the polynomials in Zagier’s work, one easily obtains new formulas for  $r(4s^2; n)$  and  $r(4s^2 + 4s; n)$  (see Corollary 2). These formulas are sums of products of divisor functions, and are simpler than those of Milne. His formulas involve Schur functions and determinants of Lambert series.

Instead of this approach, we use the fact that the map sending  $z$  to  $-1/2z$  swaps (see Proposition 2.1)  $\Theta(z)$  and the generating function for triangular numbers. Since the fundamental domain of  $\Gamma_0(2)$  has two cusps which are interchanged by this map, we obtain our formulas from Zagier’s work on the Kac–Wakimoto conjectures. This is completely elementary.

For every  $s$ , let  $A_s^\pm(\lambda)$  denote the coefficients of the polynomials

$$\prod_{i=1}^s X_i \prod_{1 \leq i < j \leq s} (X_i^2 - X_j^2)^2 = \sum_{\lambda=(a_1, \dots, a_s)} A_s^+(\lambda) X_1^{a_1} \cdots X_s^{a_s}, \quad (1.2)$$

$$\prod_{i=1}^s X_i^3 \prod_{1 \leq i < j \leq s} (X_i^2 - X_j^2)^2 = \sum_{\lambda=(a_1, \dots, a_s)} A_s^-(\lambda) X_1^{a_1} \cdots X_s^{a_s}. \quad (1.3)$$

As usual, let  $\sigma_v(n) := \sum_{d|n} d^v$ , and let  $\{B_k\}$  denote the Bernoulli numbers defined by

$$\sum_{k=0}^{\infty} B_k t^k / k! := t / (e^t - 1). \quad (1.4)$$

If  $k \geq 2$  is an even integer, then define weight  $k$  modular forms  $E^\pm(k; z)$  by

$$E^+(k; z) := 2^{2k-1} \left( \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^{4n} \right) - 2^{k-1} \left( \frac{-B_k}{2k} + \sum_{n=1}^{\infty} (-1)^n \sigma_{k-1}(n) q^n \right), \quad (1.5)$$

$$E^-(k; z) := 2^k \left( -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^{2n} \right) - \left( -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \right). \quad (1.6)$$

THEOREM 1. *If  $s$  is a positive integer, then*

$$\Theta(z)^{4s^2} = \frac{(-1)^s 4^s}{s! \prod_{j=1}^{2s-1} j!} \sum_{\lambda=(a_1, \dots, a_s)} A_s^+(\lambda) E^+(a_1 + 1; z) \cdots E^+(a_s + 1; z), \quad (1.7)$$

$$\Theta(z)^{4s^2+4s} = \frac{2^{2s^2+3s}}{s! \prod_{j=1}^{2s} j!} \sum_{\lambda=(a_1, \dots, a_s)} A_s^-(\lambda) E^-(a_1 + 1; z) \cdots E^-(a_s + 1; z). \quad (1.8)$$

COROLLARY 2. *If  $t$  is an odd integer, then define divisor functions  $\sigma_t^\pm(n)$  by*

$$\sigma_t^+(n) := \begin{cases} (2^t - 2^{2t+1}) \frac{B_{t+1}}{2t+2} & \text{if } n = 0, \\ 2^{2t+1} \sigma_t(n/4) - 2^t (-1)^n \sigma_t(n) & \text{otherwise,} \end{cases}$$

$$\sigma_t^-(n) := \begin{cases} (1 - 2^{t+1}) \frac{B_{t+1}}{2t+2} & \text{if } n = 0, \\ 2^{t+1} \sigma_t(n/2) - \sigma_t(n) & \text{otherwise.} \end{cases}$$

*If  $s$  is a positive integer, then for every non-negative integer  $n$ , we have*

$$r(4s^2; n) = \frac{(-1)^{s+n} 4^s}{s! \prod_{j=1}^{2s-1} j!} \sum_{\lambda=(a_1, \dots, a_s)} A_s^+(\lambda) \sum_{\substack{m_1 + \dots + m_s = n, \\ m_i \geq 0}} \sigma_{a_1}^+(m_1) \cdots \sigma_{a_s}^+(m_s),$$

$$r(4s^2 + 4s; n) = \frac{(-1)^n 2^{2s^2+3s}}{s! \prod_{j=1}^{2s} j!} \sum_{\lambda=(a_1, \dots, a_s)} A_s^-(\lambda) \sum_{\substack{m_1 + \dots + m_s = n, \\ m_i \geq 0}} \sigma_{a_1}^-(m_1) \cdots \sigma_{a_s}^-(m_s).$$

## 2. PROOFS

If  $k \geq 2$  is an even integer, then let  $G_k(z)$  denote the weight  $k$  Eisenstein series

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n. \quad (2.1)$$

If  $k \geq 4$ , then  $G_k$  is a weight  $k$  modular form on  $\mathrm{SL}_2(\mathbb{Z})$ . As usual, let  $\eta(z)$

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (2.2)$$

be Dedekind's eta-function. It is well known that

$$\Theta(z) = \eta^2(z)/\eta(2z). \quad (2.3)$$

Similarly, it is also well known that

$$T(z) := \frac{\eta^2(2z)}{\eta(z)} = q^{1/8} \left( \sum_{n=0}^{\infty} q^{(n^2+n)/2} \right). \quad (2.4)$$

Up to the factor  $q^{1/8}$ ,  $T(z)$  is the generating function for the triangular numbers.

**PROPOSITION 2.1.** *If  $s$  is a positive integer and  $\mathrm{Im}(z) > 0$ , then*

$$T(-1/2z)^{4s} = \frac{(-1)^s z^{2s}}{2^{2s}} \cdot \Theta(z)^{4s}.$$

*Proof.* In view of (2.3) and (2.4), the proposition follows from the fact that [2, p. 121]

$$\eta(-1/z) = \sqrt{z/i} \cdot \eta(z). \quad \blacksquare$$

**PROPOSITION 2.2.** *If  $k \geq 4$  is an even integer and  $\mathrm{Im}(z) > 0$ , then*

- (1)  $G_k(-1/4z) = (4z)^k G_k(4z)$ ,
- (2)  $G_k(-\frac{1}{4z} + \frac{1}{2}) = (2z)^k G_k(z + \frac{1}{2})$ .

*Proof.* Since  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , the modularity of  $G_k(z)$  implies

$$G_k(-1/z) = z^k G_k(z), \quad (2.5)$$

$$G_k\left(\frac{z-1}{2z-1}\right) = (2z-1)^k G_k(z). \quad (2.6)$$

Claim (1) follows from (2.5), and claim (2) follows by replacing  $z$  by  $z + \frac{1}{2}$  in (2.6). ■

PROPOSITION 2.3. *If  $\text{Im}(z) > 0$ , then*

$$(1) \quad G_2(-1/4z) = (4z)^2 G_2(4z) + \frac{24z}{\pi i}.$$

$$(2) \quad G_2(-\frac{1}{4z} + \frac{1}{2}) = (2z)^2 G_2(z + \frac{1}{2}) + \frac{24z}{\pi i}.$$

*Proof.* Let  $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  be the standard generators of  $\text{SL}_2(\mathbb{Z})$ . Claim (1) follows from the fact that [2, p. 113]

$$G_2(Sz) = G_2(-1/z) = z^2 G_2(z) + 6z/\pi i. \quad (2.7)$$

Since  $G_2(Tz) = G_2(z)$ , (2.7) and  $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = ST^{-2}S^{-1}T^{-1}$ , implies

$$G_2\left(\frac{z-1}{2z-1}\right) = (2z-1)^2 G_2(z) + 12(2z-1)/\pi i.$$

Claim (2) follows by replacing  $z$  by  $z + \frac{1}{2}$ . ■

*Proof of Theorem 1.* First we prove (1.7). If  $k \geq 2$  is even, then define  $g^+(k; z)$  by

$$g^+(k; z) := \frac{1}{2} \left( G_k(z/2) - G_k\left(\frac{z+1}{2}\right) \right). \quad (2.8)$$

Zagier [6] proved that

$$T(z)^{4s^2} = \frac{1}{4^{s(s-1)} s! \prod_{j=1}^{2s-1} j!} \sum_{\lambda=(a_1, \dots, a_s)} A_s^+(\lambda) g^+(a_1+1; z) \cdots g^+(a_s+1; z). \quad (2.9)$$

By replacing  $z$  by  $-1/2z$ , Proposition 2.1 implies

$$\begin{aligned} \Theta(z)^{4s^2} &= \frac{(-1)^s 4^s}{z^{2s^2} s! \prod_{j=1}^{2s-1} j!} \sum_{\lambda=(a_1, \dots, a_s)} A_s^+(\lambda) g^+(a_1+1; -1/2z) \\ &\quad \cdots g^+(a_s+1; -1/2z). \end{aligned} \quad (2.10)$$

By (2.8), Propositions 2.2 and 2.3, we find that

$$\begin{aligned} g^+(k; -1/2z) &= \frac{1}{2} \left( G_k(-1/4z) - G_k\left(-\frac{1}{4z} + \frac{1}{2}\right) \right) \\ &= z^k \left( 2^{2k-1} G_k(4z) - 2^{k-1} G_k\left(z + \frac{1}{2}\right) \right) = z^k E^+(k; z). \end{aligned}$$

In view of (2.10), this implies (1.7).

To prove (1.8), we begin with Zagier's formula [6]. If  $g^-(k; z) = G_k(z) - G_k(2z)$ , then

$$T(z)^{4s(s+1)} = \frac{2^s}{s! \prod_{j=1}^{2s} j!} \sum_{\lambda=(a_1, \dots, a_s)} A_s^-(\lambda) g^-(a_1 + 1; z) \cdots g^-(a_s + 1; z). \quad (2.11)$$

By (2.5), it is easy to see that

$$g^-(k; -1/2z) = 2^k z^k G_k(2z) - z^k G_k(z) = z^k E^-(k; z). \quad (2.12)$$

By Proposition 2.1, (2.11) and (2.12) implies (1.8). ■

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